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OPERATOR IDEAL NORMS ON L^p

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ABSTRACT. Let p be a real number such that $p \in (1, +\infty)$ and its conjugate exponent $q \neq 4, 6, 8 \ldots$. We prove that for an operator T defined on $L^p(\lambda)$ with values in a Banach space, the image of the unit ball determines whether T belongs to any operator ideal and its operator ideal norm. We also show that this result fails to be true in the remaining cases of p. Finally we prove that when the result holds in finite dimension, the map which associates to the image of the unit ball the operator ideal norm is continuous with respect to the Hausdorff metric.

INTRODUCTION

An important topic in the theory of linear operators is the study of operator ideals. One of the main difficulties in this theory is recognizing and giving criteria for whether an operator belongs to a certain ideal and the computation of its operator ideal norm. Some criteria have been given in terms of the image of the unit ball; for instance, a classical result in this sense is the following theorem due to A. Grothendieck (see [G] and [DU] for the definition of equimeasurable set).

Theorem. *An operator $T\colon X \longrightarrow L^1(\lambda)$ is integral if and only if it is order bounded and in this case the integral norm $i_1(T)$ satisfies $i_1(T) = \left\| \sup_{x\in B_X} |T(x)|\right\|_1$. Also, T is nuclear if and only if $T(B_X)$ is order bounded and equimeasurable.*

A necessary condition in order to characterize the belonging of an operator to a certain ideal in terms of the image of the unit ball is that this image has to determine this belonging. So, using the above result, if we have two operators $T_1\colon X_1 \longrightarrow L^1(\lambda)$ and $T_2\colon X_2 \longrightarrow L^1(\lambda)$ such that $\overline{T_1(B_{X_1})} = \overline{T_2(B_{X_2})}$, then T_1 is integral or nuclear if and only if T_2 is, where B_{X_i} denotes the closed unit ball of X_i, for $i = 1, 2$. A first question is the study of those operator ideals determined by the image of the unit ball. Some classes of operator ideals are, by definition, determined by this image, for example, compact and weakly compact operators. Unfortunately, some of the most representative classes of operator ideals lack this property. For instance, if S_1 is a quotient map from ℓ_1 onto ℓ_2, then $\overline{S_1(B_{\ell_1})} = B_{\ell_2} = S_2(B_{\ell_2})$, S_2 being the identity operator on ℓ_2. By the Grothendieck Theorem [DJT, pg. 15], S_1 is 1-summing and, of course, S_2 is not. The same example is valid for r-summing and r-integral norms for $r \in (1, +\infty)$ and can be considered with values in L^1 since ℓ_2 is isometric to a subspace of L^1.

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Therefore, if we have two operators $T_1\colon X_1 \longrightarrow Y$ and $T_2\colon X_2 \longrightarrow Y$ such that $\overline{T_1(B_{X_1})} = \overline{T_2(B_{X_2})}$, it is not always true that T_1 belongs to a given ideal if and only if T_2 does. In this paper we study certain conditions under which the image of the unit ball determines the belonging to any operator ideal. This will happen if X_1 and X_2 are L^p spaces for the same value of $p \in (1, +\infty)$ and q, the conjugate exponent of p, satisfies $q \neq 4, 6, 8, \ldots$. This will be shown in Section 1 (Theorem 1.3). The main ingredient for this result is Theorem 1.2 due to W. Lusky [Lu] which makes use of a theorem about equimeasurability of W. Rudin in [Ru2].

Results about properties of operators determined by the image of the unit ball have already been obtained in the framework of vector measures. In [R1], answering a question in [AD], it is proved that the range of a vector measure determines its total variation; that is, if two measures with values in a Banach space have the same range, then they have the same total variation. Later, in [R2], it was proved that the range also determines the Bochner derivability. Due to the correspondence between properties of vector measures and properties of integration operators (see [DU, Chapter VI]), the mentioned results can be translated into the language of operators: if λ is a finite measure and $T\colon L^\infty(\lambda) \longrightarrow X$ is a weak*-weak continuous operator, then $T(B_{L^\infty(\lambda)})$ determines the absolutely summing and nuclear norm of T. In [RR] similar results were obtained for the (r,s)-summing, r-integral and r-nuclear norms. We will see that this result holds for any operator ideal norm.

We complete Section 1 providing examples that show that these results fail to be true in the remaining cases of p. The main counterexample will be Example 1.11: if the conjugate exponent of p satisfies $q = 4, 6, 8, \ldots$, there are two operators defined on two L^p spaces, for the same value of p, one of them belonging to a certain ideal of operators and the other one not. This counterexample also disproves Theorem 1.2 but it is different from the one given in [Lu].

In Section 2 we deal with operators $T\colon L^p(\lambda) \longrightarrow \mathbb{R}^n$. If $\mathcal{K}_p$ is the collection of those sets K for which there is an operator $T\colon L^p(\lambda) \longrightarrow \mathbb{R}^n$ such that $\overline{T(B_{L^p(\lambda)})} = K$, we prove that $\mathcal{K}_p$ is closed in the set of all compact convex sets in $\mathbb{R}^n$. We finish this section showing that, for $q \neq 4, 6, 8, \ldots$, the map defined on $\mathcal{K}_p$ which associates to the image of the unit ball the operator ideal norm is continuous with respect to the Hausdorff metric.

Along the paper our notation and terminology will be consistent with [DJT]. X will be a (real, unless otherwise specified) Banach space; X^* will be its dual space. For an operator ideal $\mathcal{A}$ we denote by $\mathcal{A}(X,Y)$ the linear space of operators $T\colon X \longrightarrow Y$ in $\mathcal{A}$. If in addition $\mathcal{A}$ is a normed ideal, we denote its norm by $\|\cdot\|_{\mathcal{A}}$. $\mathcal{A}(X,Y)$ always contains the finite rank operators. Classical examples of Banach operator ideals are the ideal $[\mathcal{V}, \|\cdot\|]$ of completely continuous operators, $[\Pi_r, \pi_r]$ of r-summing operators, $[\mathcal{I}_r, i_r]$ of r-integral operators, to mention some of them (see [DJT] for the definitions).

1. Operators on $L^p(\lambda)$

In this section we will prove the announced results about operators defined on $L^p(\lambda)$. The main result is Theorem 1.3 which is a consequence of Theorem 1.2 and the following elementary observation.

Observation 1.1. *Let X_1, X_2 and Y be three Banach spaces. Two operators $T_1\colon X_1 \longrightarrow Y$ and $T_2\colon X_2 \longrightarrow Y$ satisfy $\overline{T_1(B_{X_1})} = \overline{T_2(B_{X_2})}$ if and only if, for every $y^* \in Y^*$, we have $\|T_1^* y^*\| = \|T_2^* y^*\|$.*

Indeed, being $T_1(B_{X_1})$ and $T_2(B_{X_2})$ convex sets, we have

$$\begin{aligned}\overline{T_1(B_{X_1})} = \overline{T_2(B_{X_2})} &\Leftrightarrow \text{ for every } y^* \in Y^*, \sup_{x\in B_{X_1}} \langle y^*, T_1x\rangle = \sup_{x\in B_{X_2}} \langle y^*, T_2x\rangle \\ &\Leftrightarrow \text{ for every } y^* \in Y^*, \sup_{x\in B_{X_1}} \langle T_1^*y^*, x\rangle = \sup_{x\in B_{X_2}} \langle T_2^*y^*, x\rangle \\ &\Leftrightarrow \text{ for every } y^* \in Y^*, \|T_1^*y^*\| = \|T_2^*y^*\|.\end{aligned}$$

Moreover, if for any i with $i = 1, 2$ X_i is reflexive, or X_i is a dual space and the operator T_i is weak*-weak continuous, we have $\overline{T_i(B_{X_i})} = T_i(B_{X_i})$. This will be the case we will mainly treat with the spaces L^p.

The key for our main result will be next theorem.

Theorem 1.2. *Let $1 \le q \le +\infty$, $q \ne 4, 6, 8, \dots$. Let μ, ν be two positive measures, E a subspace of $L^q(\mu)$, and $S_0\colon E \longrightarrow L^q(\nu)$ an isometry. Then there exists an extension of S_0, $S\colon L^q(\mu) \longrightarrow L^q(\nu)$ such that $\|S\| = 1$.*

For $q = \infty$ the last result is a consequence of the injectivity of $L^\infty(\nu)$, and for $q = 2$, a consequence of the fact that in a Hilbert space every subspace is complemented with a norm one projection. For the other values of q, Theorem 1.2 is due to W. Lusky [Lu, Corollary 2] for complex L^q spaces; it makes use of a result about equimeasurability of W. Rudin [Ru2, Theorem 1] for complex functions. The real version of this result was given by W. Linde in [L] after the appearance of the paper of Lusky and it shows that Theorem 1.2 holds for real L^q spaces.

We are now ready to state our main result. In what follows q will be the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. In next result, when we say that $T\colon L^\infty(\lambda) \longrightarrow X$ is weak*-weak continuous, we assume that $L^\infty(\lambda)$ is the dual of $L^1(\lambda)$ (which is always true out of pathological examples); so, this condition means that $T^*(X^*) \subseteq L^1(\lambda)$.

Theorem 1.3. *Let $p \in (1, +\infty]$ such that $q \ne 4, 6, 8, \dots$, $\mathcal{A}$ an operator ideal and X a Banach space. If $T_1\colon L^p(\lambda_1) \longrightarrow X$ and $T_2\colon L^p(\lambda_2) \longrightarrow X$ are two operators, weak*-weak continuous if $p = +\infty$, such that*

$$T_1(B_{L^p(\lambda_1)}) = T_2(B_{L^p(\lambda_2)});$$

then $T_1 \in \mathcal{A}(L^p(\lambda_1), X)$ if and only if $T_2 \in \mathcal{A}(L^p(\lambda_2), X)$. If in addition $\mathcal{A}$ is a normed operator ideal, then $\|T_1\|_{\mathcal{A}} = \|T_2\|_{\mathcal{A}}$.

Proof. Consider $T_1^*\colon X^* \longrightarrow L^q(\lambda_1)$ and $T_2^*\colon X^* \longrightarrow L^q(\lambda_2)$ such that $T_1(B_{L^p(\lambda_1)}) = T_2(B_{L^p(\lambda_2)})$. By Observation 1.1, for every $x^* \in X^*$ we have $\|T_1^*x^*\| = \|T_2^*x^*\|$. Taking $E = T_1^*(X^*)$ and the map $T_1^*x^* \mapsto T_2^*x^*$, we have an isometry $S_0\colon E \longrightarrow L^q(\lambda_2)$ that can be extended to $S\colon L^q(\lambda_1) \longrightarrow L^q(\lambda_2)$ by Theorem 1.2. Then $T_2^* = S \circ T_1^*$ and, therefore, $T_2 = T_1 \circ S^*$ with $\|S^*\| \le 1$. This implies that, if $\mathcal{A}$ is an operator ideal such that $T_1 \in \mathcal{A}(L^p(\lambda_1), X)$, then $T_2 \in \mathcal{A}(L^p(\lambda_2), X)$. If $\mathcal{A}$ is a normed ideal, then $\|T_2\|_{\mathcal{A}} \le \|T_1\|_{\mathcal{A}}$. In a similar way we obtain, if $T_2 \in \mathcal{A}(L^p(\lambda_2), X)$, that $T_1 \in \mathcal{A}(L^p(\lambda_1), X)$ and $\|T_1\|_{\mathcal{A}} \le \|T_2\|_{\mathcal{A}}$. □

Although Theorem 1.2 holds for $q = \infty$, Theorem 1.3 does not hold for $p = 1$, as we will see in Example 1.6. When we try to reproduce the proof, if $T_1\colon L^1(\lambda_1) \to X$ and $T_2\colon L^1(\lambda_2) \to X$ satisfy $\overline{T_1(B_{L^1(\lambda_1)})} = \overline{T_2(B_{L^1(\lambda_2)})}$, we obtain that $\|T_1^{**}\|_{\mathcal{A}} = \|T_2^{**}\|_{\mathcal{A}}$. The same happens if $p = \infty$ and we drop the condition of being weak*-weak continuous operators. Or even, more generally, for operators defined on $\mathcal{C}(K)$ spaces since their duals are L^1 spaces. Using Theorem 1.2 for $q = 1$, we have that, if $T_1\colon \mathcal{C}(K_1) \to X$ and $T_2\colon \mathcal{C}(K_2) \to X$ satisfy $\overline{T_1(B_{\mathcal{C}(K_1)})} = \overline{T_2(B_{\mathcal{C}(K_2)})}$, then $\|T_1^{**}\|_{\mathcal{A}} = \|T_2^{**}\|_{\mathcal{A}}$.

We will also see, in Example 1.7, that Theorem 1.3 does not hold for $\mathcal{C}(K)$ spaces or even for weakly compact operators defined on L^∞ (see Example 1.8). However, the following lemma and the observation we have just made show that Theorem 1.3 holds for $p=1$ and $\mathcal{C}(K)$ spaces when we restrict ourselves to operators valued in a finite dimensional space, a fact that we will isolate in Corollary 1.5.

Lemma 1.4. *Let $Y = \mathcal{C}(K)$ or $Y = L^1(\lambda)$. For an operator $T\colon Y \longrightarrow \mathbb{R}^n$, and any normed operator ideal $\mathcal{A}$, we have $\|T^{**}\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}$.*

Proof. It is clear that $\|T\|_{\mathcal{A}} \le \|T^{**}\|_{\mathcal{A}}$, since $T = T^{**} \circ k_Y$, $k_Y\colon Y \longrightarrow Y^{**}$ being the canonical injection.

To prove the reverse inequality, we will use that Y^* has the metric approximation property. Let $\varphi_j \in Y^*$, $j = 1, \dots, n$, be elements such that $Ty = (\langle y, \varphi_j\rangle)_{j=1}^n$. Then, for every $\varepsilon > 0$, there exists an operator $Q\colon Y^* \longrightarrow Y^*$ with finite rank such that $Q\varphi_j = \varphi_j$ and $\|Q\| \le 1+\varepsilon$ (see [P, 10.2.4]). Also, $Q^*\colon Y^{**} \longrightarrow Y^{**}$ is a finite rank operator. Put $E = Q^*(Y^{**})$. By the local reflexivity Principle [DJT, pg. 178], for every $\varepsilon > 0$ there exists $R\colon E \longrightarrow Y$ such that $\|R\| \le 1+\varepsilon$, and $\langle Ry^{**}, \varphi_j\rangle = \langle y^{**}, \varphi_j\rangle$, for every $y^{**} \in E$ and $j = 1, \dots, n$. Then we have, for every $y^{**} \in Y^{**}$,

$$\begin{aligned} T^{**}y^{**} &= (\langle \varphi_j, y^{**}\rangle)_j = (\langle Q\varphi_j, y^{**}\rangle)_j \\ &= (\langle \varphi_j, Q^* y^{**}\rangle)_j = (\langle \varphi_j, R\circ Q^* y^{**}\rangle)_j = T\circ R \circ Q^* y^{**}. \end{aligned}$$

From this we deduce $\|T^{**}\|_{\mathcal{A}} \le \|T\|_{\mathcal{A}}(1+\varepsilon)^2$ for every $\varepsilon > 0$; so, $\|T^{**}\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}$. □

Corollary 1.5. *Take $Y_1 = L^1(\lambda_1)$ and $Y_2 = L^1(\lambda_2)$, or $Y_1 = \mathcal{C}(K_1)$ and $Y_2 = \mathcal{C}(K_2)$. Fix a norm on $\mathbb{R}^n$; if $T_1\colon Y_1 \to \mathbb{R}^n$ and $T_2\colon Y_2 \to \mathbb{R}^n$ are two operators such that $\overline{T_1(B_{Y_1})} = \overline{T_2(B_{Y_2})}$, and $\mathcal{A}$ is a normed operator ideal, then $\|T_1\|_{\mathcal{A}} = \|T_2\|_{\mathcal{A}}$.*

Remark. Assume that the ideal of operators $\mathcal{A}$ has finite dimensional nature, that is, for every $T\colon Y \longrightarrow X$,

$$\|T\|_{\mathcal{A}} = \sup_q \|q\circ T\|_{\mathcal{A}},$$

where the supremum runs over every map $q\colon X \longrightarrow Z$ with Z a finite dimensional space and $\|q\| \le 1$; then last result holds for any Banach space X. This is the case of maximal ideals as (r,s)-summing and r-integral operators.

Example 1.6. There are two operators $T_1\colon L^1[0,1] \longrightarrow L^1[0,1]$ and $T_2\colon \ell_1 \longrightarrow L^1[0,1]$ such that $\overline{T_1(B_{L^1[0,1]})} = \overline{T_2(B_{\ell_1})}$ but T_2 is completely continuous while T_1 is not.

It is enough to consider the identity operator T_1 on $L^1[0,1]$ and T_2 a quotient operator from ℓ_1 onto $L^1[0,1]$. By the Schur property of ℓ_1, T_2 is completely continuous (takes weakly compact sets to norm compact sets); but T_1 is not, since we can find weakly compact sets in $L^1[0,1]$ that are not compact.

Example 1.7. There are two compact sets K_1 and K_2, two operators $T_1\colon \mathcal{C}(K_1) \to \ell_2$ and $T_2\colon \mathcal{C}(K_2) \to \ell_2$, and an operator ideal $\mathcal{A}$ such that $\overline{T_1(B_{\mathcal{C}(K_1)})} = \overline{T_2(B_{\mathcal{C}(K_2)})}$, $T_1 \in \mathcal{A}$ but $T_2 \notin \mathcal{A}$.

Consider the Cantor group $\Delta = \{-1,1\}^{\mathbb{N}}$. Then it is easy to see that the class of all operators which factor through $\mathcal{C}(\Delta)$ forms an operator ideal $\mathcal{A}$. Let $\{r_n\}_n$ be the sequence of Rademacher functions on Δ (the coordinate functions $r_n\big((\varepsilon_k)_{k=1}^{\infty}\big) = \varepsilon_n$), μ the Haar measure on Δ (the product probability considering on each factor $\{-1,1\}$ the one which assigns $1/2$ to every point), and the operator $T_1\colon \mathcal{C}(\Delta) \longrightarrow \ell_2$ defined by

$$T_1(h) = \Big(\int h r_n \, d\mu\Big)_n, \qquad \text{for every } h \in \mathcal{C}(\Delta).$$

Let $T_2\colon L^\infty(\mu) \longrightarrow \ell_2$ be the operator defined by

$$T_2(g) = \Big(\int g r_n \, d\mu\Big)_n, \qquad \text{for every } g \in L^\infty(\mu).$$

Since $B_{\mathcal{C}(\Delta)}$ is weak* dense in $B_{L^\infty(\mu)}$, and T_2 is weak*-weak continuous, we have that $\overline{T_1(B_{\mathcal{C}(\Delta)})} = T_2(B_{L^\infty(\mu)})$. One must remember that $L^\infty(\mu)$ is isometric to $\mathcal{C}(K_2)$ for certain compact set K_2.

It is obvious that $T_1 \in \mathcal{A}$. We have to show that $T_2 \notin \mathcal{A}$. If T_2 factors through $\mathcal{C}(\Delta)$, we could find $A\colon L^\infty(\mu) \longrightarrow \mathcal{C}(\Delta)$ and $B\colon \mathcal{C}(\Delta) \longrightarrow \ell_2$ such that $T_2 = B \circ A$. The operator A is weakly compact since $\mathcal{C}(\Delta)$ is separable and does not contain a copy of ℓ_∞ (see [DU, Corollary VI.1.3]). B is completely continuous as it is weakly compact and $\mathcal{C}(\Delta)$ has the Dunford-Pettis property. This would force $T_2 = B \circ A$ to be compact, a contradiction.

Example 1.8. Two operators $T_1\colon L^\infty(\mu_1) \to \ell_2(\mathbb{R})$ and $T_2\colon L^\infty(\mu_2) \to \ell_2(\mathbb{R})$, and an operator ideal $\mathcal{A}$ such that $\overline{T_1(B_{L^\infty(\mu_1)})} = T_2(B_{L^\infty(\mu_2)})$, $T_1 \in \mathcal{A}$, but $T_2 \notin \mathcal{A}$.

It is a known fact that the non separable space $\ell_1(\mathbb{R})$ is isometric to a subspace of ℓ_∞. There exists a quotient operator from $\ell_1(\mathbb{R})$ onto the non separable Hilbert space $\ell_2(\mathbb{R})$. By the Grothendieck Theorem, this quotient operator is 1-summing, then 2-summing, and so it has a 2-summing extension to an operator defined on ℓ_∞. Therefore, there exists a surjective operator $T_1\colon \ell_\infty \to \ell_2(\mathbb{R})$ which is 2-summing, and such that $B_{\ell_2(R)} \subseteq T_1(B_{\ell_\infty})$.

We will take as $L^\infty(\mu_1) = \ell_\infty$, and the ideal $\mathcal{A}$ of operators which factor through ℓ_∞. Evidently T_1 belongs to this ideal. If we consider now ℓ_∞ as the space $\mathcal{C}(\beta\mathbb{N})$, by the Pietsch Factorization Theorem, there exist a Radon probability μ_2 on $\beta\mathbb{N}$, and an operator $S\colon L^2(\mu_2) \to \ell_2(\mathbb{R})$ such that $T_1\psi = S\psi$, for every $\psi \in \mathcal{C}(\beta\mathbb{N})$. Our operator T_2 will be the restriction of S to $L^\infty(\mu_2)$ which is weak*-weak continuous. As in the previous example, we have

$$\overline{T_1(B_{L^\infty(\mu_1)})} = \overline{T_1(B_{\mathcal{C}(\beta N)})} = T_2(B_{L^\infty(\mu_2)})\,.$$

It remains to show that T_2 does not factor through ℓ_∞; this will be the most arduous task. If we suppose that $T_2 = A \circ B$ with $B\colon L^\infty(\mu_2) \to \ell_\infty$, and $A\colon \ell_\infty \to \ell_2(\mathbb{R})$, there exists a sequence $(x_n^*)_n$ in $L^\infty(\mu_2)^*$ with $Bh = \big(x_n^*(h)\big)_n$, for every $h \in L^\infty(\mu_2)$. We will show that we can find $h_0 \in L^\infty(\mu_2)$, with $T_2 h_0 \neq 0$, and $x_n^*(h_0) = 0$, for every $n \in \mathbb{N}$. This would contradict that $T_2 = A \circ B$.

Each x_n^* can be decomposed in the form $x_n^* = f_n + y_n^*$, with $f_n \in L^1(\mu_2)$ and y_n^* a purely finitely additive functional, which satisfies that, for every $\varepsilon_n > 0$, there exists a measurable set D_n with $\mu_2(D_n) \geq 1 - \varepsilon_n$, and $y_n^*\big(h\chi_{D_n}\big) = 0$, for every $h \in L^\infty(\mu_2)$. Taking $\varepsilon_n = \varepsilon/2^n$, for certain $\varepsilon > 0$, we could find the same measurable set D, with $\mu_2(D) \geq 1 - \varepsilon$, and $y_n^*\big(h\chi_D\big) = 0$, for each $n \in \mathbb{N}$ and every $h \in L^\infty(\mu_2)$.

There exists a bounded family $(g_t)_{t\in\mathbb{R}}$ in $L^2(\mu_2)$ such that

$$T_2 h = \Big(\int h g_t \, d\mu_2\Big)_{t\in\mathbb{R}}, \qquad \text{for every } h \in L^\infty(\mu_2)\,.$$

Since $B_{\ell_2(R)} \subseteq T_1(B_{\ell_\infty})$, we have, for $s \neq t$, that $\|g_t - g_s\|_1 \geq 1$. Using the Cauchy–Schwartz inequality, we have

$$\|\chi_D g_t - g_t\|_1 \leq \|\chi_D - 1\|_2 \|g_t\|_2 \leq \sqrt{\varepsilon} M\,.$$

If ε is small enough, we have, for $s, t \in \mathbb{R}$, $s \neq t$,

$$\|\chi_D g_s - \chi_D g_t\|_1 \geq 1/2\,.$$

The closed linear space Y generated by the sequence $(f_n \chi_D)$ in $L^1(\mu_2)$ is separable, so it cannot contain all the functions $\chi_D g_t$. There exists $t_0 \in \mathbb{R}$, such that $\chi_D g_{t_0}$ does not belong to Y; by the Hahn-Banach Theorem, there exists $h_1 \in L^\infty(\mu_2)$ such that

$$\int h_1 \chi_D g_{t_0} \, d\mu_2 \neq 0, \qquad \text{and} \qquad \int h_1 \chi_D f_n \, d\mu_2 = 0, \qquad \text{for every } n \in \mathbb{N}.$$

The function $h_0 = h_1 \chi_D$ is the one we are looking for, since, for every $n \in \mathbb{N}$,

$$x_n^*(h_0) = \int f_n h_0 \, d\mu_2 + y_n^*\big(h_1 \chi_D\big) = 0.$$

In [Lu] an example is given to show that, for $q = 4, 6, 8, \ldots$, an isometry as in Theorem 1.2 does not necessarily extend to a contraction on $L^q(\mu)$. We give in Example 1.11 an isometry that is not extendable even as a bounded operator. This example also disproves Theorem 1.3 for those values of p with conjugate exponent $q = 4, 6, 8, \ldots$. To this end, we will construct two operators $S_1 \colon Y \longrightarrow L^q(\lambda_1)$ and $S_2 \colon Y \longrightarrow L^q(\lambda_2)$ such that $\|S_1(y)\| = \|S_2(y)\|$, for every $y \in Y$; but S_2 is q-integral while S_1 is not. Let us remark that if the isometry $S_2 y \mapsto S_1 y$ were extendable to a bounded operator, then S_1 would be q-integral too. Considering the adjoint operators $T_1 = S_1^*$ and $T_2 = S_2^*$, we have that T_2 belongs to the dual ideal [DJT, pg. 186] of q-integral operators, that is, the ideal of those operators whose adjoint is q-integral, while T_1 does not (let us recall that an operator is q-integral precisely when its second adjoint is [DJT, pg. 104]). In [P, 19.1.4], the dual ideal of q-integral operators is identified with the $(q, 1, q)$-integral operators. Since $\|S_1(y)\| = \|S_2(y)\|$ for every $y \in Y$, thanks to the weak*-weak continuity of T_1 and T_2, following Observation 1.1 we have $T_1(B_{L^p(\lambda_1)}) = T_2(B_{L^p(\lambda_2)})$.

We will use harmonic analysis in our example. We denote by $\mathbb{T}$ the torus, the multiplicative group of complex numbers of modulus 1, and by $\mathbb{T}_r$ ($r \in \mathbb{N}$) the subgroup of r-roots of unity in $\mathbb{T}$. In $\mathbb{T}$ and $\mathbb{T}_r$ we consider their (normalized) Haar measures. We will need the following lemma.

Lemma 1.9. *Let q be a real number such that $q = 2k$ ($k > 1$ an integer) and p its conjugate exponent. Then for infinitely many positive integers r there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\|1 + \alpha \cos t + \beta \sin t\|_{L^p(\mathbb{T}_r)} > \|1 + \alpha \cos t + \beta \sin t\|_{L^p(\mathbb{T})}.$$

Proof. Suppose that for certain r we have, for every $\alpha, \beta \in \mathbb{R}$,

$$\|1 + \alpha \cos t + \beta \sin t\|_{L^p(\mathbb{T})} \geq \|1 + \alpha \cos t + \beta \sin t\|_{L^p(\mathbb{T}_r)}.$$

That is,

$$\frac{1}{2\pi}\int_0^{2\pi} |1+\alpha\cos t+\beta\sin t|^p\,dt \geq \frac{1}{r}\sum_{j=0}^{r-1}\Big|1+\alpha\cos(\frac{2\pi j}{r})+\beta\sin(\frac{2\pi j}{r})\Big|^p.$$

Taking $\alpha^2+\beta^2=1$ the last inequality turns, for every $u\in\mathbb{R}$, to

$$\varphi(u)=\frac{1}{2\pi}\int_0^{2\pi}\big(1+\cos(t+u)\big)^p\,dt \geq \frac{1}{r}\sum_{j=0}^{r-1}\big(1+\cos(\frac{2\pi j}{r}+u)\big)^p=\psi_r(u).$$

$\varphi(u)$ is a constant function on u due to the translation invariance of the Haar measure on $\mathbb{T}$. This would lead to the fact that $\psi_r(u)$ is also constant since ψ_r and φ have the same mean on $[0,2\pi]$. Observe that $1<p<2$, so $\rho(u)=(1+\cos u)^p=\sum_{k=0}^{\infty}a_k\cos(ku)$ is not a $\mathcal{C}^\infty$ function and therefore is not a trigonometric polynomial. We obtain that there exist infinitely many k such that $a_k\neq 0$. Taking those r such that $a_r\neq 0$, we have that $\psi_r(u)=\sum_{k=0}^{\infty}a_{rk}\cos(rku)$ is not a constant function. The lemma follows. □

Throughout Example 1.11 we will assume all scalars to be complex. This is more suitable for using harmonic analysis; however a real example of this situation can be given (see the remark following this example).

Let G be a compact abelian group. We denote by Γ its dual group, that is,

$$\Gamma=\{\gamma\colon G\longrightarrow\mathbb{T}:\ \gamma \text{ is a continuous homomorphism}\}.$$

Elements of Γ are called characters of the group G. We will use the following examples:

i) $G=\mathbb{T}$. Γ can be identified with $\mathbb{Z}$; for every $m\in\mathbb{Z}$, we consider the character γ_m on $\mathbb{T}$ defined by $\gamma_m(z)=z^m$ for every $z\in\mathbb{T}$.

ii) For N a positive integer, $G=\mathbb{T}^N$. Γ can be identified with $\mathbb{Z}^N$ considering for any $\vec{m}=(m_n)_{n=1}^N\in\mathbb{Z}^N$ the character $\gamma_{\vec{m}}((z_n)_{n=1}^N)=\prod_{n=1}^N z_n^{m_n}$.

iii) $G=\mathbb{T}^{\mathbb{N}}$. In this case its dual group can be identified with $\Gamma=\mathbb{Z}^{(\mathbb{N})}$ the sequences of $\mathbb{Z}$ which are eventually null. For every $\vec{m}=(m_n)_{n=1}^\infty\in\mathbb{Z}^{(\mathbb{N})}$, we associate the character $\gamma_{\vec{m}}((z_n)_{n=1}^\infty)=\prod_{n=1}^\infty z_n^{m_n}$. Observe that, except for a finite number of n's, we have $z_n^{m_n}=1$.

Let m_G be the Haar probability measure on G. We denote by $L^q(G)$ the space $L^q(m_G)$. Observe that Γ is a subset of $L^q(G)$ whose linear span is dense in $L^q(G)$; the same happens with $\mathcal{C}(G)$. If Λ is a subset of Γ, we denote by $\mathcal{C}_\Lambda(G)$ and $L^q_\Lambda(G)$ the closed linear span of the elements of Λ in $\mathcal{C}(G)$ and $L^q(G)$, respectively.

Let G_1 be a closed subgroup of G. Every character of G, when restricted to G_1, defines a character of G_1; but different elements in Γ can define the same character on G_1. In our examples we will consider G_1 to be $\mathbb{T}_r$, $\mathbb{T}_r^N$ and $\mathbb{T}_r^{\mathbb{N}}$, respectively, and Λ to be:

i) If $G=\mathbb{T}$, $\Lambda=\Lambda_1=\{-1,0,1\}\subset\mathbb{Z}$.

ii) If $G=\mathbb{T}^N$, $\Lambda=\Lambda_N=\{(m_n)\in\mathbb{Z}^N:|m_n|\leq 1,\ n=1,2,\dots,N\}$.

iii) If $G=\mathbb{T}^{\mathbb{N}}$, $\Lambda=\Lambda_\infty=\{(m_n)\in\mathbb{Z}^{(\mathbb{N})}:|m_n|\leq 1,\ \text{for all } n\in\mathbb{N}\}$.

It is easy to see that if $r\geq 3$, different elements of Λ, when restricted to G_1, produce different characters on G_1. So we can consider $\mathcal{C}_\Lambda(G_1)$ and $L^q_\Lambda(G_1)$. The following lemma will be used in Example 1.11.

Lemma 1.10. *Consider G, G_1 and Λ as in the previous examples. Then, if $q = 2k$ for an integer $k > 1$, and $r \geq q + 1$, we have, for every eventually null complex sequence $(a_\gamma)_{\gamma\in\Lambda}$,*

$$\Big\| \sum_{\gamma\in\Lambda} a_\gamma \gamma \Big\|_{L^q(G)} = \Big\| \sum_{\gamma\in\Lambda} a_\gamma \gamma \Big\|_{L^q(G_1)}.$$

Proof. Suppose that $G = \mathbb{T}^{\mathbb{N}}$. Observe that

$$\big| \sum_{\gamma\in\Lambda} a_\gamma \gamma \big|^q = \Big| \big(\sum_{\gamma\in\Lambda} a_\gamma \gamma \big)^k \Big|^2 = \Big| \sum_{\delta\in\Delta} b_\delta \delta \Big|^2$$

where Δ is the set of characters corresponding to $\{(m_n) \in \mathbb{Z}^{(\mathbb{N})} : |m_n| \leq k,$ for all $n \in \mathbb{N}\}$. Thanks to the condition $r \geq 2k + 1$, the elements in Δ produce different characters on G_1. By the orthogonality of the set of characters with respect to the Haar measure we have

$$\begin{aligned} \Big\| \sum_{\gamma\in\Lambda} a_\gamma \gamma \Big\|^q_{L^q(G)} &= \Big\| \sum_{\delta\in\Delta} b_\delta \delta \Big\|^2_{L^2(G)} \\ &= \sum_{\delta\in\Delta} |b_\delta|^2 = \Big\| \sum_{\delta\in\Delta} b_\delta \delta \Big\|^2_{L^2(G_1)} = \Big\| \sum_{\gamma\in\Lambda} a_\gamma \gamma \Big\|^q_{L^q(G_1)}. \end{aligned}$$

□

Example 1.11. If $q = 2k$, for an integer $k > 1$, there exist two probability measures λ_1 and λ_2, a Banach space Y and two operators $S_1 \colon Y \longrightarrow L^q(\lambda_1)$ and $S_2 \colon Y \longrightarrow L^q(\lambda_2)$ such that $\|S_1(y)\| = \|S_2(y)\|$, for every $y \in Y$. S_2 is q-integral and S_1 is not. Consequently, Theorem 1.3 does not hold for $q = 4, 6, 8, \ldots$.

Proof. The last sentence in the statement was explained before Lemma 1.9. The probability measure λ_1 will be the Haar measure on $G = \mathbb{T}^{(\mathbb{N})}$, and λ_2 the Haar measure on $G_1 = \mathbb{T}_r^{(\mathbb{N})}$. Y will be the space $\mathcal{C}_\Lambda(G_1)$ with $\Lambda = \Lambda_\infty$ as in Lemma 1.10. S_2 is the inclusion of $\mathcal{C}_\Lambda(G_1)$ in $L^q(G_1)$; as it factors through the natural inclusion maps $\mathcal{C}_\Lambda(G_1) \to \mathcal{C}(G_1) \to L^q(G_1)$, S_2 is q-integral and $i_q(S_2) = 1$.

Let us consider the operator $S_1 \colon \mathcal{C}_\Lambda(G_1) \to L^q(G)$ defined as $S_1 = j_1 \circ i \circ i_1$ for

$$\mathcal{C}_\Lambda(G_1) \xrightarrow{i_1} L^q_\Lambda(G_1) \xrightarrow{i} L^q_\Lambda(G) \xrightarrow{j_1} L^q(G)$$

where i_1, j_1 are the natural inclusion maps, and i is the isometry established in Lemma 1.10 sending $\gamma\,|_{G_1}$ to γ, for every $\gamma \in \Lambda$. By construction, it is easy to see that $\|S_1(y)\| = \|S_2(y)\|$, for every $y \in Y$.

In order to see that S_1 is not q-integral we use an averaging argument similar to the one introduced by W. Rudin in [Ru1] and used by A. Pełczynski to provide an example of a q-summing operator which is not q-integral (see [Pe]). To avoid measurability justifications we will work in $\mathbb{T}^N$ and $\mathbb{T}_r^N$. If S_1 were q-integral with q-integral norm $i_q(S_1) = M$, then, for every N, the analogous operator $S_1^N \colon \mathcal{C}_{\Lambda_N}(\mathbb{T}_r^N) \to L^q(\mathbb{T}^N)$ would satisfy $i_q(S_1^N) \leq M$: considering each function on $\mathbb{T}_r^N$ as a function on $\mathbb{T}_r^{\mathbb{N}}$ depending on the first N coordinates, there is a natural isometrical injection $i_N \colon \mathcal{C}_{\Lambda_N}(\mathbb{T}_r^N) \to \mathcal{C}_{\Lambda_\infty}(\mathbb{T}_r^{\mathbb{N}})$; taking a conditional expectation over the first N coordinates there exists a norm one projection $p_N \colon L^q(\mathbb{T}^{\mathbb{N}}) \to L^q(\mathbb{T}^N)$; and we have $S_1^N = p_N \circ S_1 \circ i_N$.

So let $G = \mathbb{T}^N$, $G_1 = \mathbb{T}_r^N$ and $\Lambda = \Lambda_N$, and assume that $i_q(S_1^N) \leq M$. This implies [DJT, Prop. 6.12] that there exists an operator $S \colon \mathcal{C}(G_1) \to L^q(G)$ such that

$i_q(S) = i_q(S_1^N) \le M$, and S extends S_1^N, that is, $Sf = S_1^N f$ for every $f \in \mathcal{C}_\Lambda(G_1)$. Using additive notation for the operation on G, for every $x \in G_1$ and every function f defined on G or G_1, let $\tau_x(f) = f_x$, the function given by $f_x(t) = f(t-x)$. τ_x is an isometry on $\mathcal{C}_\Lambda(G_1)$, $\mathcal{C}(G_1)$ and $L^q(G)$. For $x \in G_1$, let $H_x = \tau_{-x} \circ S \circ \tau_x$. Then $i_q(H_x) \le \|\tau_{-x}\| i_q(S) \|\tau_x\| \le M$. For every $f \in \mathcal{C}_\Lambda(G_1)$, $f = \sum_{\gamma\in\Lambda} a_\gamma \gamma$,

$$H_x(f) = H_x\Big(\sum_{\gamma\in\Lambda} a_\gamma\gamma\Big) = \sum_{\gamma\in\Lambda} a_\gamma H_x(\gamma) = \sum_{\gamma\in\Lambda} a_\gamma\gamma = S_1^N(f).$$

Now observe that G_1 is a finite group. Define the operator $H\colon \mathcal{C}(G_1) \to L^q(G)$ as the average of the operators H_x, that is,

$$H(f) = \frac{1}{|G_1|}\sum_{x\in G_1} H_x(f) = \int_{G_1} H_x(f) dm_{G_1}(x), \qquad \text{for } f \in \mathcal{C}(G_1).$$

We still have $i_q(H) \le M$ and H is an extension of S_1^N. Even more, H is translation invariant under elements of G_1, that is, for $t \in G_1$,

$$\begin{aligned} H(f_t) &= \int_{G_1} (\tau_{-x} \circ S \circ \tau_x)(f_t)\, dm_{G_1}(x) \\ &= \tau_t \int_{G_1} (\tau_{-t-x} \circ S \circ \tau_{x+t})(f)\, dm_{G_1}(x) = (Hf)_t. \end{aligned}$$

Since $i_q(H) \le M$, there exists a probability measure μ on G_1 such that

$$\|Hf\|_{L^q(G)} \le M\|f\|_{L^q(\mu)} \qquad \text{for every } f \in \mathcal{C}(G_1).$$

Let us see that we can take as μ the Haar measure on G_1: for each $x \in G_1$, $\|Hf\|_{L^q(G)} = \|(Hf)_x\|_{L^q(G)} = \|H(f_x)\|_{L^q(G)} \le M\|f_x\|_{L^q(\mu)}$; thus

$$\|Hf\|^q_{L^q(G)} \le M^q \int_{G_1} \|f_x\|^q_{L^q(\mu)}\, dm_{G_1}(x) = M^q \int_{G_1} \Big(\int_{G_1} |f_x(t)|^q\, d\mu(t)\Big)\, dm_{G_1}(x)$$

$$= M^q \int_{G_1} \Big(\int_{G_1} |f_x(t)|^q\, dm_{G_1}(x)\Big)\, d\mu(t) = M^q \int \|f\|^q_{L^q(G_1)}\, d\mu = M^q \|f\|^q_{L^q(G_1)}.$$

So last inequality allows us to consider H as an operator $H\colon L^q(G_1) \to L^q(G)$ which is invariant under translation in G_1, $\|H\| \le M$, and $H\gamma = \gamma$ for every $\gamma \in \Lambda$.

We are going to prove that for every $g \in L^q(G_1)$ and every $\gamma \in \Lambda$ we have

$$\int_G Hg(x)\gamma(-x)\, dm_G(x) = \int_{G_1} g(x)\gamma(-x)\, dm_{G_1}(x). \tag{1}$$

By linearity we only have to prove (1) for g a character of G_1. If $g = \gamma\,|_{G_1}$, then $Hg = \gamma$ and both terms in (1) are equal to one. Otherwise, there exists $t \in G_1$ such that $g(t) \ne \gamma(t)$. Being g a character, we have $g_t(x) = g(x-t) = g(x)g(-t)$ for every $x \in G_1$, and therefore $Hg_t = g(-t)Hg$. Using $Hg_t = (Hg)_t$, we have

$$\begin{aligned} g(-t)\int_G Hg(x)\gamma(-x)\, dm_G(x) &= \int_G (Hg)_t(x)\gamma(-x)\, dm_G(x) \\ &= \int_G Hg(x-t)\gamma(-x+t)\gamma(-t)\, dm_G(x) = \gamma(-t)\int_G Hg(z)\gamma(-z)\, dm_G(z) \end{aligned}$$

This implies $\int_G Hg(x)\gamma(-x)\, dm_G(x) = 0$. As we also have $\int_{G_1} g(x)\gamma(-x)\, dm_{G_1}(x) = 0$, the proof of (1) is finished.

Let $(b_\gamma)_{\gamma\in\Lambda}$ be complex numbers and $f = \sum_{\gamma\in\Lambda} b_\gamma\gamma$. We can choose $g \in L^q(G_1)$ with $\|g\|_{L^q(G_1)} = 1$ such that

$$\|f\|_{L^p(G_1)} = \int_{G_1} g(x)f(-x)\,dm_{G_1}(x).$$

Using (1) and linearity we have

$$(2)\quad \|f\|_{L^p(G_1)} = \int_G Hg(x)f(-x)\,dm_G(x) \le \|Hg\|_{L^q(G)}\|f\|_{L^p(G)} \le M\|f\|_{L^p(G)}.$$

By Lemma 1.9, we can choose r such that there is a function $f_1\colon \mathbb{T} \longrightarrow \mathbb{C}$, $f_1 \in L^p_{\Lambda_1}(\mathbb{T})$ with $\|f_1\|_{L^p(\mathbb{T})} = 1$ and $\|f_1\|_{L^p(\mathbb{T}_r)} = \eta > 1$. Define now $f_N\colon \mathbb{T} \longrightarrow \mathbb{C}$ by $f_N(z_1,\dots,z_N) = f_1(z_1)\cdots f_1(z_N)$. It is easy to see that $f_N \in L^p_{\Lambda_N}(\mathbb{T}^N)$. As the Haar measure on $\mathbb{T}^N$ ($\mathbb{T}^N_r$) is the product of N-times the Haar measure on $\mathbb{T}$ ($\mathbb{T}_r$) we have $\|f_N\|_{L^p(\mathbb{T}^N)} = 1$ and $\|f_N\|_{L^p(\mathbb{T}^N_r)} = \eta^N$. So by (2), we have $M \ge \eta^N$ for every N, which is a contradiction as $\eta > 1$. □

Remark. Although in last example scalars were complex, let us observe that all the Banach spaces considered in it are Banach spaces of functions closed under conjugation. Taking the real Banach spaces of either the real part or the imaginary part of those functions, we could get a counterexample to Theorem 1.3 over the field of real numbers.

2. Continuity of the operator ideal norms

Fix $p \in [1,+\infty]$. We are going to study in this section operators defined on L^p spaces with values in a finite dimensional space. For $p \in (1,+\infty]$ we will see that we can reduce our study to operators defined on $L^p(\mu)$, for μ a finite measure on the Euclidean unit sphere. We will show that, if $q \ne 4,6,8,\dots$, the map which associates to the image of the unit ball its operator ideal norm is continuous with respect to the Hausdorff metric.

Let $\mathcal{K}$ be the collection of all non empty compact and convex subsets in $\mathbb{R}^n$. $\mathcal{K}$ is a complete metric space with the Hausdorff metric d defined for $K_1, K_2 \in \mathcal{K}$ by

$$d(K_1,K_2) = \inf\{\varepsilon : K_1 \subseteq K_2 + \varepsilon B_{\ell_2^n}, K_2 \subseteq K_1 + \varepsilon B_{\ell_2^n}\}.$$

The *support function* of $K \in \mathcal{K}$ is defined for $\xi \in \mathbb{R}^n$ as

$$\Psi_K(\xi) = \sup\{\langle x,\xi\rangle : x \in K\}.$$

We denote by $\mathcal{K}_p$ the subset of $\mathcal{K}$ of those sets K for which there is an operator $T\colon L^p(\lambda) \longrightarrow \mathbb{R}^n$ so that $\overline{T(B_{L^p(\lambda)})} = K$. For $p = 1$, $\mathcal{K}_1$ coincides with the symmetric elements of $\mathcal{K}$. $\mathcal{K}_2$ is the set of ellipsoids in $\mathbb{R}^n$. For $p = \infty$, $\mathcal{K}_p$ coincides with the set of zonoids in $\mathbb{R}^n$ symmetric over 0, and with the set of those $K \in \mathcal{K}$, for which there are a compact Hausdorff Ω and an operator T from $\mathcal{C}(\Omega)$ to $\mathbb{R}^n$ with $\overline{T(B_{\mathcal{C}(\Omega)})} = K$. In fact, if $K = \overline{T(B_{\mathcal{C}(\Omega)})}$, there exist a Radon measure μ on Ω, and $f_1,\dots,f_n$ in $L^1(\mu)$ such that

$$T(g) = \Big(\int g f_k\,d\mu\Big)_{k=1}^n, \qquad \text{for every } g \in \mathcal{C}(\Omega);$$

since $B_{\mathcal{C}(\Omega)}$ is weak* dense in $B_{L^\infty(\mu)}$, and the operator T has a trivial weak* continuous extension $\tilde{T}\colon L^\infty(\mu) \to \mathbb{R}^n$, K is a zonoid which coincides with $\tilde{T}(B_{L^\infty(\mu)})$.

When $p \in (1,+\infty]$ there is an analogous characterization of the elements of $\mathcal{K}_p$ to the one given in [B] for zonoids. For every positive finite measure μ on the

Euclidean unit sphere $\mathbb{S}^{n-1}$, we will denote by S_μ^p, or just S_μ when there is no possible confussion, the operator $S_\mu^p\colon L^p(\mu) \longrightarrow \mathbb{R}^n$ defined by

$$S_\mu^p(\psi) = \int \psi \cdot \vec{x}\, d\mu(\vec{x}), \qquad \text{for every } \psi \in L^p(\mu).$$

We will write $K_\mu = S_\mu(B_{L^p(\mu)})$.

Proposition 2.1. *Let $p \in (1, +\infty]$. If $K \in \mathcal{K}_p$, there exists a positive finite symmetric measure μ on $\mathbb{S}^{n-1}$ such that $K = K_\mu = S_\mu(B_{L^p(\mu)})$.*

Proof. Consider an operator $T\colon L^p(\lambda) \longrightarrow \mathbb{R}^n$ for $p \in (1, +\infty]$ (weak* continuous if $p = +\infty$) such that $K = T(B_{L^p(\lambda)})$. Then there exist n functions $f_1, \dots, f_n$ in $L^q(\lambda)$ such that

$$T(\varphi) = \int \varphi(f_1, \dots, f_n)\, d\lambda$$

for every function $\varphi \in L^p(\lambda)$. Put $f = (f_1, \dots, f_n)$ and $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$; take

$$h = \frac{f}{\|f\|} \qquad (\text{with } \frac{0}{0} = 0).$$

Since every f_i is a function in $L^q(\lambda)$, the measure ν with density $\|f\|^q$ with respect to λ is finite. Then h is $\mathbb{S}^{n-1}$-valued ν-a.e. Let μ_T be the image measure of ν by the function h on the Borel sets of $\mathbb{S}^{n-1}$.

The operator S_{μ_T} satisfies, for every $\xi \in \mathbb{R}^n$, $T^*\xi = \langle \xi, f\rangle$ and $S_{\mu_T}^*\xi = \xi$. Since

$$\begin{aligned}\|\xi\|_{L^q(\mu_T)}^q = \int |\langle \xi, \vec{x}\rangle|^q\, d\mu_T(\vec{x}) &= \int |\langle \xi, h\rangle|^q\, d\nu \\ &= \int |\langle \xi, f\rangle|^q\, d\lambda = \|T^*\xi\|_{L^q(\lambda)}^q,\end{aligned}$$

we obtain, using Observation 1.1, that $K = T(B_{L^p(\lambda)}) = S_{\mu_T}(B_{L^p(\mu_T)})$.

For every Borel set A in $\mathbb{S}^{n-1}$ let $\check{\mu}_T$ be the measure $\check{\mu}_T(A) = \mu_T(-A)$. Let us denote by μ the symmetrization of μ_T, that is, $\mu = \frac{1}{2}(\mu_T + \check{\mu}_T)$. It is clear that $K = S_{\mu_T}(B_{L^p(\mu_T)}) = S_\mu(B_{L^p(\mu)})$ since $\|S_{\mu_T}^*\xi\| = \|\xi\|_{L^q(\mu_T)} = \|\xi\|_{L^q(\mu)} = \|S_\mu^*\xi\|$ for every $\xi \in \mathbb{R}^n$. □

Let us denote by $\mathcal{M}$ the closed cone of all positive finite and symmetric measures on $\mathbb{S}^{n-1}$ as a subset of $\mathcal{C}(\mathbb{S}^{n-1})^*$ with the weak* topology. If $\mu \in \mathcal{M}$, let us define $H_p(\mu) = K_\mu = S_\mu(B_{L^p(\mu)})$. We have a map H_p defined on $\mathcal{M}$ with values in $\mathcal{K}_p$. The following lemma can be found in [B] for $p = \infty$; we have adapted its proof for the general case $p \in (1, \infty]$.

Lemma 2.2. *Suppose that $p \in (1, +\infty]$. Then $H_p\colon (\mathcal{M}, w^*) \longrightarrow (\mathcal{K}_p, d)$ is continuous.*

Proof. Suppose that the net μ_α converges to μ in the weak* topology of $\mathcal{M}$. We will show that K_{μ_α} converges to K_μ. It is enough to show that there is a subnet μ_γ for which K_{μ_γ} converges to K_μ. Since μ_α converges to μ in the weak* topology, we have that $\|\mu_\alpha\| = \int 1\, d\mu_\alpha$ converges to $\|\mu\| = \int 1\, d\mu$. Therefore, taking a subnet if necessary, we can assume that $\{\|\mu_\alpha\|\}$ is bounded. Let Ψ_{μ_α} be the support function of K_{μ_α} and $\rho_\alpha = \sup\{\|x\| : x \in K_\alpha\}$. It is clear that $\rho_\alpha = \sup\{\Psi_{\mu_\alpha}(\xi) : \|\xi\| = 1\}$. Since

$$\Psi_{\mu_\alpha}(\xi) = \Big(\int |\langle x, \xi\rangle|^q \, d\mu_\alpha(x)\Big)^{\frac{1}{q}},$$

we have that $\rho_\alpha \le (\mu_\alpha(\mathbb{S}^{n-1}))^{1/q}$. So $\{\rho_\alpha\}$ is bounded and using Blaschke selection principle [DJT, pg. 425], there exist a subnet μ_γ and a set $K \in \mathcal{K}$ so that K_{μ_γ} converges to K. Then Ψ_{μ_γ} converges pointwise to Ψ_K. For every $\xi \in \mathbb{R}^n$, the function $h_\xi(x) = |\langle x, \xi\rangle|^q$ is a continuous function on $\mathbb{S}^{n-1}$, so

$$\Psi_{\mu_\gamma}(\xi) = \Big(\int |\langle x, \xi\rangle|^q \, d\mu_\gamma(x)\Big)^{\frac{1}{q}} \to \Big(\int |\langle x, \xi\rangle|^q \, d\mu(x)\Big)^{\frac{1}{q}} = \Psi_\mu(\xi)$$

and we conclude $\Psi_\mu = \Psi_K$; so, $K = K_\mu$. This proves that the map H_p is continuous. □

We are going to use last result to prove that $\mathcal{K}_p$ is closed in $\mathcal{K}$. We will also need the following lemma.

Lemma 2.3. *Suppose that $p \in (1, +\infty]$. If $K_\mu \in \mathcal{K}_p$ satisfies $K_\mu \subseteq rB_{\ell_2^n}$, then*

$$\|\mu\| \le n^{q/2} n r^q.$$

Proof. Suppose that $K_\mu \subseteq rB_{\ell_2^n}$. Then, using that $\|\vec{x}\|_2 \le n^{1/2}\|\vec{x}\|_\infty \le n^{1/2}\|\vec{x}\|_q$ for $\vec{x} \in \mathbb{R}^n$,

$$\begin{aligned}\|\mu\| &= \int_{S^{n-1}} \|\vec{x}\|_2 \, d\mu(\vec{x}) \le n^{1/2} \int_{S^{n-1}} \|\vec{x}\|_q \, d\mu(\vec{x}) \\ &\le n^{1/2}\Big(\int_{S^{n-1}} \|\vec{x}\|_q^q \, d\mu(\vec{x})\Big)^{1/q} \|\mu\|^{1/p},\end{aligned}$$

the last inequality by Hölder's inequality. Then,

$$\|\mu\| \le n^{q/2} \sum_{i=1}^{n} \int_{S^{n-1}} |\langle \vec{x}, e_i\rangle|^q \, d\mu(\vec{x}) \le n^{q/2} n r^q.$$

□

Proposition 2.4. *For $p \in [1, +\infty]$, $\mathcal{K}_p$ is closed in $\mathcal{K}$.*

Proof. It is obvious for $p = 1$. Take now $p \in (1, +\infty]$. Suppose that $\{K_{\mu_n}\}$ is a Cauchy sequence in $\mathcal{K}_p$. Then, using previous lemma, $\{\|\mu_n\|\}$ is bounded and there is a subsequence $\{\mu_{n_j}\}$ converging to μ in $\mathcal{M}$. Lemma 2.2 implies that $K_{\mu_{n_j}}$ converges to $K_\mu \in \mathcal{K}_p$, so K_{μ_n} also converges to K_μ, an element of $\mathcal{K}_p$. □

We will make use of the following result, whose proof can be found in [K], [N] and [L].

Theorem 2.5. *Let q be a real number in $[1, +\infty)$ which is not an even integer. If μ and ν are two finite, positive and symmetric measures on $\mathbb{S}^{n-1}$ such that, for every $\xi \in \mathbb{R}^n$,*

$$\int |\langle x, \xi\rangle|^q \, d\mu(x) = \int |\langle x, \xi\rangle|^q \, d\nu(x);$$

then $\mu = \nu$.

It is important to note that this result does not hold when q is an even integer, since in this case, the space generated by the functions $|\langle \cdot, \xi\rangle|^q$, with $\xi \in \mathbb{R}^n$, in the space of continuous and symmetric functions on the sphere is finite dimensional.

Lemma 2.6. *Let $p \in (1, +\infty]$ such that $q \neq 2, 4, 6, \ldots$. If $K \in \mathcal{K}_p$, there exists a unique positive finite symmetric measure μ on $\mathbb{S}^{n-1}$ such that $K = K_\mu$.*

Proof. Suppose that $K = K_\mu = K_\nu$ where μ and ν are two finite, positive and symmetric measures on $\mathbb{S}^{n-1}$. By Observation 1.1, μ and ν satisfy the hypothesis of previous theorem. This implies that $\mu = \nu$. □

Remark. If we have an operator $T\colon L^p(\lambda) \longrightarrow \mathbb{R}^n$ with $q \neq 2, 4, 6, 8, ...$, let μ_T be the measure obtained in the proof of Proposition 2.1. One can prove that for any normed operator ideal $\mathcal{A}$, $\|T\|_{\mathcal{A}} = \|S_{\mu_T}\|_{\mathcal{A}}$; in this case, the factorization argument given in Theorem 1.3 is much simpler. Considering the isometries $R\colon L^p(\mu_T) \longrightarrow L^p(\lambda)$ defined by $R(\psi) = (\psi \circ h)\|f\|^{q-1}$ and $Q\colon L^q(\mu_T) \longrightarrow L^q(\lambda)$ defined by $Q(\psi) = (\psi \circ h) \cdot \|f\|$, one can easily show that $S_{\mu_T} = T \circ R$ and $T = S_{\mu_T} \circ Q^*$. So $\|T\|_{\mathcal{A}} = \|S_{\mu_T}\|_{\mathcal{A}}$. Later it is not difficult to show that $\|S_\mu\|_{\mathcal{A}} = \|S_{\mu_T}\|_{\mathcal{A}}$ where μ is the symmetrization of the measure μ_T. As μ does not depend on the operator T, but only on $T(B_{L^p(\lambda)})$, this provides a new proof in this case of Theorem 1.3. We have isolated the proof in the finite dimensional case since it is more explicit and clarifies the ideas used in the general case hidden behind the proof of Theorem 1.2. Also, the usage of measures on $\mathbb{S}^{n-1}$ will be useful in what follows and provides an idea of the reason why Theorem 1.3 fails for $q = 4, 6, 8, \ldots$.

Proposition 2.7 . *Suppose that $p \in (1, +\infty]$ is a real number such that $q \neq 2, 4, 6, \ldots$. Then $H_p\colon (\mathcal{M}, w^*) \longrightarrow (\mathcal{K}_p, d)$ is a homeomorphism.*

Proof. Using Lemma 2.2 and Lemma 2.6 we know that the map H_p which associates to every μ the set K_μ is continuous and injective.

Now take $K_0 \in \mathcal{K}_p$. If we prove that there exist $C > 0$ and a neighborhood V of K_0 in $\mathcal{K}_p$ such that $\|H_p^{-1}(K)\| \leq C$ for every $K \in V$, we will have that H_p establishes a homeomorphism between the compact set $\{\mu \in \mathcal{M} : \|\mu\| \leq C\}$ and a compact set in $\mathcal{K}_p$ which contains K_0 in its interior; H_p^{-1} will be continuous at K_0. Take $r > 0$ such that $K_0 \subseteq rB_{\ell_2^n}$. The set $V = \{K \in \mathcal{K}_p : K \subseteq (r+1)B_{\ell_2^n}\}$ is a neighborhood of K_0. If $K \in V$ and $\mu = H_p^{-1}(K)$, using Lemma 2.3

$$\|\mu\| \leq n^{q/2} n (r+1)^q.$$

□

Let $\mathcal{A}$ be a normed operator ideal; Theorem 1.3 lets us define, for $p \in [1, +\infty]$, with $q \neq 4, 6, 8, \ldots$, the map $\|\cdot\|_{\mathcal{A}}\colon \mathcal{K}_p \to \mathbb{R}$ as $\|K\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}$, for an operator $T\colon L^p(\lambda) \to \mathbb{R}^n$ such that $\overline{T(B_{L^p(\lambda)})} = K$. We are going to show that this map is continuous with respect to the Hausdorff metric. For $p \neq 1, 2$ we use the homeomorphism given in previous result. We begin with two lemmas in which we isolate the cases $p = 1$ and $p = 2$.

Lemma 2.8. *Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$ and a normed operator ideal $\mathcal{A}$; the map $\|\cdot\|_{\mathcal{A}}\colon \mathcal{K}_1 \longrightarrow \mathbb{R}$ is continuous for the Hausdorff metric.*

Proof. Consider $K_1, K_2 \in \mathcal{K}_1$ such that $d(K_1, K_2) \leq \delta$. Take two sequences $\{x_n\}$ in K_1, $\{y_n\}$ in K_2 such that $\{x_n\}$ is dense in K_1 and $\{y_n\}$ is dense in K_2. Let $\{x'_n\}$ be a sequence in K_2 such that $\|x_n - x'_n\| \leq \delta$, $\{y'_n\}$ in K_1 with $\|y_n - y'_n\| \leq \delta$. If

$T_i\colon \ell_1 \longrightarrow \mathbb{R}^n$, $i = 1, 2$, are two operators such that $T_1 e_{2n-1} = x_n$ and $T_1 e_{2n} = y'_n$, $T_2 e_{2n} = y_n$ and $T_2 e_{2n-1} = x'_n$, with $(e_n)_n$ the canonical basis in ℓ_1, then we have $K_1 = \overline{T_1(B_{\ell_1})}$, $K_2 = \overline{T_2(B_{\ell_1})}$ and $\|T_1 - T_2\| \le \delta$. Let $C > 0$ be a constant such that $\|T\|_{\mathcal{A}} \le C\|T\|$ for every operator $T\colon \ell_1 \longrightarrow \mathbb{R}^n$; it is enough to take as C the norm in $\mathcal{A}$ of the identity operator in $\mathbb{R}^n$ with the fixed norm. Then

$$\big|\|T_1\|_{\mathcal{A}} - \|T_2\|_{\mathcal{A}}\big| \le \|T_1 - T_2\|_{\mathcal{A}} \le C\|T_1 - T_2\| \le C\delta,$$

and we obtain the continuity. □

Lemma 2.9. *Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$ and a normed operator ideal $\mathcal{A}$; the map $\|\cdot\|_{\mathcal{A}}\colon \mathcal{K}_2 \longrightarrow \mathbb{R}$ is continuous for the Hausdorff metric.*

Proof. Suppose first that $E_0 \in \mathcal{K}_2$ is an n-dimensional ellipsoid. Then $E_0 = T(B_{\ell_2^n})$ for certain one-to-one operator defined on ℓ_2^n with values on $\mathbb{R}^n$. Since the metric d is equivalent to d' defined for $K_1, K_2 \in \mathcal{K}$ by

$$d'(K_1, K_2) = \inf\{\varepsilon : K_1 \subseteq K_2 + \varepsilon E_0, K_2 \subseteq K_1 + \varepsilon E_0\},$$

a neighborhood V of E_0 for the Hausdorff metric in $\mathcal{K}_2$ is one of the form $V = \{E \in \mathcal{K}_2 : \frac{1}{\tau}E_0 \subseteq E \subseteq \tau E_0\}$, with $\tau > 1$. If $E \in V$, there exists a one-to-one operator S on ℓ_2^n with values on $\mathbb{R}^n$ such that $S(B_{\ell_2^n}) = E$. Consider $R = S^{-1} \circ T$. Then we have that $\|R\|, \|R^{-1}\| \le \tau$. Since $S \circ R = T$ and $T \circ R^{-1} = S$, we obtain $\|T\|_{\mathcal{A}} \le \tau\|S\|_{\mathcal{A}}$ and $\|S\|_{\mathcal{A}} \le \tau\|T\|_{\mathcal{A}}$. Taking τ close enough to 1, we obtain the continuity of $\|\cdot\|_{\mathcal{A}}$ at E_0.

To prove the lemma in the general case, let Y be the linear span generated by E_0 and P_Y the orthogonal projection over Y. If $d(E, E_0) \le \delta$, then $d(\vec{x}, Y) \le \delta$ for every $\vec{x} \in E$. If $E = S(B_{\ell_2^n})$, we have

$$\big\|S\vec{t} - P_Y \circ S\vec{t}\big\| < \delta \text{ for every } \vec{t} \in B_{\ell_2^n}.$$

Then $\big|\|S\|_{\mathcal{A}} - \|P_Y \circ S\|_{\mathcal{A}}\big| \le \|S - P_Y \circ S\|_{\mathcal{A}} \le C\|S - P_Y \circ S\| \le C \cdot \delta$. On the other hand, $d(P_Y(E), E_0) \le \delta$. This reduces this case to the previous one; so given $\varepsilon > 0$, we can choose δ small enough such that $\big|\|E_0\|_{\mathcal{A}} - \|P_Y(E)\|_{\mathcal{A}}\big| \le \varepsilon$ and $C\delta \le \varepsilon$. We conclude that $\big|\|E_0\|_{\mathcal{A}} - \|E\|_{\mathcal{A}}\big| \le 2\varepsilon$ if δ is small enough. This finishes the proof. □

To prove the continuity in the rest of the cases we will use following lemma.

Lemma 2.10. *Let μ be a positive finite measure on $\mathbb{S}^{n-1}$. Then for every $\varepsilon > 0$ there exists a partition π of $\mathbb{S}^{n-1}$ such that $\pi = \{A_i\}_{i=1}^k \cup \{F_i\}_{i=1}^k$ where every A_i is open for $i = 1, \dots, k$, $\operatorname{diam}(E) < \varepsilon$ for every $E \in \pi$, and $\mu(F_i) = 0$ for $i = 1, \dots, k$.*

Proof. For every $\varepsilon > 0$, we can find a finite set $\{\vec{x}_i\}_{i=1}^k$ in $\mathbb{S}^{n-1}$ such that $\mathbb{S}^{n-1} = \bigcup_{i=1}^k B(\vec{x}_i, \frac{\varepsilon}{4})$. Put $S_r(x_i) = \{\vec{x} \in \mathbb{S}^{n-1} : \|\vec{x}_i - \vec{x}\| = r\}$. Since μ is a finite measure, there exist sets of the form $S_{r_i}(x_i)$ with $r_i \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}]$ such that $\mu(S_{r_i}) = 0$, for $i = 1, \dots, k$. Take $A_1 = B(x_1, r_1)$ and $F_1 = S_{r_1}(x_1)$. Once we have found $A_1, \dots, A_i$ and $F_1, \dots, F_i$ for $i = 1, \dots, k-1$, we take $A_{i+1} = B(x_{i+1}, r_{i+1}) \setminus \big(\bigcup_{j=1}^i \overline{B(\vec{x}_j, r_j)}\big)$ and $F_{i+1} = S_{r_{i+1}}(x_{i+1}) \setminus \bigcup_{j=1}^i \overline{B(\vec{x}_j, r_j)}$. It is obvious that these sets satisfy all the conclusions in our lemma. □

Theorem 2.11. *Fix a norm on $\mathbb{R}^n$, a normed operator ideal $\mathcal{A}$, and $p \in [1, +\infty]$ such that $q \neq 4, 6, \dots$. The map $\|\cdot\|_{\mathcal{A}}\colon \mathcal{K}_p \longrightarrow \mathbb{R}$ is continuous for the Hausdorff metric.*

Proof. The cases $p = 1$ and $p = 2$ have already been proved. Suppose now that $p \in (1, +\infty]$ and $q \neq 2, 4, 6, \ldots$. By virtue of Proposition 2.7, it is enough to prove that the map $N_{\mathcal{A}} \colon \mu \mapsto \|S^p_\mu\|_{\mathcal{A}}$ is continuous from $\mathcal{M}$ to $\mathbb{R}$. We can find a constant $C > 0$ such that, for every Banach space X, and every $T \colon X \to \mathbb{R}^n$, we have $\|T\|_{\mathcal{A}} \leq C\|T\|$, where $\|T\|$ denotes the norm as an operator from X to ℓ_2^n; it is enough to take as C the norm on $\mathcal{A}$ of the identity on ℓ_2^n with values in $\mathbb{R}^n$ with the fixed norm.

To show the continuity of $N_{\mathcal{A}}$ at $\mu = 0$, it is enough to observe that $\|S^p_\nu\| \leq \nu(\mathbb{S}^{n-1})^{1/q}$ as in the proof of Lemma 2.2; then

$$N_{\mathcal{A}}(\nu) \leq C\|\nu\|^{1/q} < \varepsilon$$

if $\|\nu\|$ is small enough, which is possible in some neighborhood of $\mu = 0$ in $\mathcal{M}$.

Suppose, then, that $\mu \neq 0$ and we will prove that $N_{\mathcal{A}}$ is continuous at μ. Fix $\delta > 0$, and for $\varepsilon = \delta^2/2$, we take a partition π of $\mathbb{S}^{n-1}$ as in Lemma 2.10. Put $F = \bigcup_{i=1}^k F_i$ and $A = \bigcup_{i=1}^k A_i$. We choose $\varepsilon_i > 0$, for $i = 1, \ldots, k$, such that, if $\mu(A_i) > 0$, then $0 < \varepsilon_i < \mu(A_i)/2$, and

$$\sigma = \sum_{i=1}^k \varepsilon_i \leq \min\left\{\frac{1}{2}\delta^2, \frac{1}{2}\delta \min\{\mu(A_i) : \mu(A_i) > 0\}\right\}.$$

Since for every open set G in $\mathbb{S}^{n-1}$ we have

$$\mu(G) = \sup\left\{ \int \varphi\, d\mu : \varphi \in \mathcal{C}(\mathbb{S}^{n-1}), \quad 0 \leq \varphi \leq \chi_G \right\},$$

and F is a closed set with $\mu(F) = 0$, we can find a neighborhood U_δ of μ in $\mathcal{M}$ in such a way that, for every $\nu \in U_\delta$, the following three conditions hold:

(a) $\nu(A_i) > \mu(A_i) - \varepsilon_i$, $i = 1, \ldots, k$;
(b) $\nu(F) < \varepsilon$;
(c) $|\nu(\mathbb{S}^{n-1}) - \mu(\mathbb{S}^{n-1})| = \left| \int 1\, d\mu - \int 1\, d\nu \right| < \varepsilon$.

Then, whenever $\mu(A_i) > 0$,

$$\frac{\mu(A_i)}{\nu(A_i)} \leq 1 + \frac{\varepsilon_i}{\nu(A_i)} \leq 1 + \frac{2\varepsilon_i}{\mu(A_i)} \leq 1 + \frac{2\sigma}{\mu(A_i)} \leq 1 + \delta. \tag{3}$$

For $\nu \in U_\delta$, let E_ν and E_μ be the conditional expectations on $L^q(\nu)$ and $L^q(\mu)$, respectively, with respect to the σ-algebra generated by the partition π. Let $X_\nu = E_\nu(L^q(\nu))$ and $X_\mu = E_\mu(L^q(\mu))$. We define $R_\nu \colon X_\nu \longrightarrow X_\mu$ for $\psi \in X_\nu$, by $R_\nu(\psi) = \psi$. It is easy to prove, because of (3), that $\|R_\nu\| \leq (1+\delta)^{\frac{1}{q}}$.

Put $I = \{i \in \{1, \ldots, k\} : \mu(A_i)(1+\delta) > \nu(A_i)\}$. Then we have

$$\begin{aligned}
\mu(\mathbb{S}^{n-1}) + \varepsilon \geq \nu(\mathbb{S}^{n-1}) &\geq \sum_{i=1}^k \nu(A_i) \geq \sum_{i \in I} \nu(A_i) + (1+\delta) \sum_{i \notin I} \mu(A_i) \\
&\geq \sum_{i \in I} \mu(A_i) - \sum_{i=1}^k \varepsilon_i + \sum_{i \notin I} \mu(A_i) + \delta \mu(\bigcup_{i \notin I} A_i) \\
&= \mu(\mathbb{S}^{n-1}) - \sigma + \delta \mu(\bigcup_{i \notin I} A_i).
\end{aligned}$$

So we have

$$\mu(\bigcup_{i\notin I} A_i) \le \frac{\varepsilon+\sigma}{\delta} \le \delta.$$

We also obtain from before

$$\begin{aligned}\nu(\bigcup_{i\in I} A_i) &\ge \mu(\bigcup_{i\in I} A_i) - \sigma = \mu(\mathbb{S}^{n-1}) - \mu(\bigcup_{i\notin I} A_i) - \sigma \\ &\ge \nu(\mathbb{S}^{n-1}) - \varepsilon - \delta - \sigma \ge \nu(\mathbb{S}^{n-1}) - \delta^2 - \delta.\end{aligned}$$

That is,

$$\nu(\bigcup_{i\notin I} A_i) \le \delta + \delta^2.$$

Let $\widetilde{A} = \bigcup_{i\in I} A_i$ and $R_\mu \colon X_\mu \longrightarrow X_\nu$ defined for $\psi \in X_\mu$, by $R_\mu(\psi) = \psi\chi_{\widetilde{A}}$. It is then clear, by choice of the set I, that $\|R_\mu\| \le (1+\delta)^{\frac{1}{q}}$.

Let I_μ be the injection of X_μ into $L^q(\mu)$, and I_ν the injection of X_ν into $L^q(\nu)$. We have the following diagram:

$$\begin{array}{ccccccc} \mathbb{R}^n & \xrightarrow{(S^p_\nu)^*} & L^q(\nu) & \xrightarrow{E_\nu} & X_\nu & \xrightarrow{I_\nu} & L^q(\nu) \\ & & & & R_\nu \big\downarrow \big\uparrow R_\mu & & \\ \mathbb{R}^n & \xrightarrow[(S^p_\mu)^*]{} & L^q(\mu) & \xrightarrow[E_\mu]{} & X_\mu & \xrightarrow[I_\mu]{} & L^q(\mu) \end{array}$$

Since $\operatorname{diam}(E) < \varepsilon$ for every $E \in \pi$, we have $\big\|\xi - E_\nu\xi\big\|_{L^\infty(\nu)} \le \varepsilon\|\xi\|$ for $\xi \in \mathbb{R}^n$. Since $\nu(A_i) = 0$ implies $\mu(A_i) = 0$, and $R_\nu(\chi_{A_i}) = \chi_{A_i}$, we also have

$$\big\|(\xi - R_\nu \circ E_\nu\xi)\chi_{A_i}\big\|_{L^\infty(\mu)} \le \varepsilon\|\xi\|.$$

For every $\xi \in \mathbb{S}^{n-1}$, we obtain

$$\begin{aligned}\big\|(S^p_\mu)^*\xi - (I_\mu \circ R_\nu \circ E_\nu \circ (S^p_\nu)^*)\xi\big\|^q_{L^q(\mu)} &= \big\|\xi - (I_\mu \circ R_\nu \circ E_\nu)\xi\big\|^q_{L^q(\mu)} \\ &= \big\|\xi - (R_\nu \circ E_\nu)\xi\big\|^q_{L^q(\mu)} = \sum_{i=1}^k \int_{A_i} \big|\xi - R_\nu \circ E_\nu)\xi\big|^q \, d\mu \\ &\le \varepsilon^q\mu(A) = \varepsilon^q\mu(\mathbb{S}^{n-1})\,;\end{aligned}$$

i.e., $\big\|(S^p_\mu)^* - (I_\mu \circ R_\nu \circ E_\nu \circ (S^p_\nu)^*)\big\| \le \varepsilon\mu(\mathbb{S}^{n-1})^{\frac{1}{q}}$. Then $\big\|S^p_\mu - S^p_\nu \circ E^*_\nu \circ R^*_\nu \circ I^*_\mu\big\|_{\mathcal{A}} \le C\varepsilon\mu(\mathbb{S}^{n-1})^{\frac{1}{q}}$, and using the properties of normed ideal,

$$N_{\mathcal{A}}(\mu) \le C\varepsilon\mu(\mathbb{S}^{n-1})^{\frac{1}{q}} + (1+\delta)^{\frac{1}{q}} N_{\mathcal{A}}(\nu) \le C'\delta + (1+\delta)^{\frac{1}{q}} N_{\mathcal{A}}(\nu). \tag{4}$$

Conversely, if $i \in I$ and then $\mu(A_i) > 0$, we have, for every $\xi \in \mathbb{R}^n$,

$$\big\|(\xi - R_\mu \circ E_\nu\xi)\chi_{A_i}\big\|_{L^\infty(\nu)} \le \varepsilon\|\xi\|.$$

We obtain now, for every $\xi \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\big\|(S^p_\nu)^*\xi - (I_\nu \circ R_\mu \circ E_\mu \circ (S^p_\mu)^*)\xi\big\|^q_{L^q(\nu)} &= \big\|\xi - (R_\mu \circ E_\mu)\xi\big\|^q_{L^q(\nu)} \\ = \sum_{i\in I}\int_{A_i} \|\xi - (R_\mu \circ E_\mu)\xi\|^q \, d\nu &+ \sum_{i\notin I}\int_{A_i} \|\xi\|^q \, d\nu + \sum_{i=1}^k \int_{F_i} \|\xi\|^q \, d\nu \\ \le \varepsilon^q\nu(\widetilde{A}) + \nu(A \setminus \widetilde{A}) + \nu(F) &\le \varepsilon^q(\mu(\mathbb{S}^{n-1}) + \varepsilon) + (\delta + \delta^2) + \varepsilon.\end{aligned}$$

We deduce, analogously to the previous case,

$$N_{\mathcal{A}}(\nu) \leq \eta(\delta) + (1+\delta)^{\frac{1}{q}} N_{\mathcal{A}}(\mu), \tag{5}$$

with $\eta(\delta) \to 0^+$ when $\delta \to 0^+$. From (4) and (5) we deduce the continuity of $N_{\mathcal{A}}$ at μ. □

References

[AD] Anantharaman, R. and Diestel, J., *Sequences in the range of a vector measure*, Comment. Math. Prace Mat. **30** (1991), 221–235. MR **92g:**46049

[B] Bolker, E. D., *A class of convex bodies*, Trans. Amer. Math. Soc. **145** (1969), 323–345. MR **41:**921

[DJT] Diestel, J., Jarchow, H. and Tonge, A., *Absolutely summing operators*, Cambridge studies in advance mathematics 43, Cambridge, 1995. MR **96i:**46001

[DU] Diestel, J. and Uhl Jr., J.J., *Vector Measures*, Amer. Math. Soc. Surveys 15, Providence, R. I., 1977. MR **56:**12216

[G] Grothendieck, A., *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955). MR **17:**763c

[K] Kanter, M., *The L_p-norm of sums of translates of a function*, Trans. Amer. Math. Soc. **179** (1973), 35–47. MR **50:**14062

[L] Linde, D.R., *Moments of measures in Banach spaces*, Math. Ann. **258** (1982), 277–287. MR **83f:**60010

[Lu] Lusky, W., *Some consequences on Rudin's paper "L^p-isometries and equimeasurability"*, Indiana University Math. J. **27** (1978), 859–866. MR **81h:**46028

[N] Neyman, A., *Representation of L_p-norms and isometric embedding in L_p-spaces*, Israel J. Math. **48** (1984), 129–138. MR **86g:**46033

[Pe] Pełczynski, A., *p-integral operators commuting with group representations and examples of quasi-p-integral operators which are not p-integral*, Studia Math. **33** (1969), 63–70. MR **39:**6125

[P] Pietsch, A., *Operator ideals*, North-Holland, Amsterdam, 1980. MR **81j:**47001

[R1] Rodríguez-Piazza, L., *The range of a vector measure determines its total variation*, Proc. Amer. Math. Soc. **111** (1991), 205–214. MR **91e:**46053

[R2] Rodríguez-Piazza, L., *Derivability, variation and range of a vector measure*, Studia Math. **112** (1995), 165–187. MR **96c:**28014

[RR] Rodríguez-Piazza, L. and Romero-Moreno, M.C., *Conical measures and properties of a vector measure determined by its range*, Studia Math. 125 (1997), 255–270. MR **98i:**46042

[Ru1] Rudin, W., *Projections on invariant subspaces*, Proc. Amer. Math. Soc. **13** (1962), 215–228. MR **25:**1460

[Ru2] Rudin, W., *L^p-isometries and equimeasurability*, Indiana University Math. J. **25** (1976), 429–432. MR **53:**14105

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